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# Domination parameters of strong product of directed paths

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**Abstract**

In a digraph $D = (V(D), A(D))$, a vertex $u$ dominates a vertex $v$ if $u = v$ or $uv \in A(D)$. A set $S \subseteq V(D)$ is a dominating set of $D$ if any vertex of $V(D)$ is dominated by at least one vertex of $S$. The domination number of $D$, denoted by $\gamma(D)$, is the minimum cardinality of a dominating set of $D$. A two-valued function $f : V(D) \to \{-1, 1\}$ defined on the vertices of $D$ is called a signed dominating function if $f(N^-[v]) \geq 1$ for every $v$ in $D$. The weight of a signed dominating function is $f(V(D)) = \sum_{v\in V(D)} f(v)$. The minimum weight of a signed dominating function of $D$ is the signed domination number $\gamma_s(D)$ of $D$. Let $\vec{P}_m \boxtimes \vec{P}_n$ be the strong product of directed paths $\vec{P}_m$ and $\vec{P}_n$. In this paper, we determine the exact values of $\gamma(\vec{P}_m \boxtimes \vec{P}_n)$ and $\gamma_s(\vec{P}_m \boxtimes \vec{P}_n)$ for any $m$ and $n$, respectively.

**MSC:** 05C69

**Keywords:** Domination number, Signed domination number, Strong product, Directed path.

## 1 Introduction

All digraphs considered in this paper are finite without loops and multiple arcs. For notation and graph-theoretical terminology not defined here we follow [3]. In a

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digraph $D$ with a vertex set $V(D)$ and an arc set $A(D)$, $u$ is said to be an *in-neighbor* of $v$ if $uv \in A(D)$. For a vertex $v \in V(D)$, denote $N_D^-(v) = \{u : uv \in A(D)\}$ be the *open in-neighborhood* of $v$. The *closed in-neighborhood* of $v$ is $N_D^-[v] = N_D^-(v) \cup \{v\}$. In all cases above, we omit the subscript $D$ when the digraph $D$ is clear from the context. For $S \subseteq V(D)$, $D[S]$ denotes the subdigraph induced by $S$.

For two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$, the *strong product* $D_1 \boxtimes D_2$ is the digraph with vertex set $V_1 \times V_2$ and $(x_1, x_2)(y_1, y_2) \in A(D_1 \times D_2)$ if and only if $x_1 = y_1$ and $x_2y_2 \in A_2$ or $x_2 = y_2$ and $x_1y_1 \in A_1$ or $x_1y_1 \in A_1$ and $x_2y_2 \in A_2$, where $x_i, y_i \in V_i$ for $i = 1, 2$. We use $D_1 \cong D_2$ to present that $D_1$ and $D_2$ are isomorphic. Throughout this paper, we denote the sets of vertices of directed paths $\vec{P}_m$ and $\vec{P}_n$ by $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively. Furthermore, in strong product $\vec{P}_m \boxtimes \vec{P}_n$ (see Figure 1), let $X_j = \bigcup_{i=1}^{n}\{(u_j, v_i)\}$ for $1 \le j \le m$ and let $Y_i = \bigcup_{j=1}^{m}\{(u_j, v_i)\}$ for $1 \le i \le n$.

Now domination for undirected graphs is well studied and surveyed in [7]. However, domination for directed graphs (digraphs) has not yet been investigated extensively. A set $S \subseteq V(D)$ is a dominating set of the digraph $D$ if every vertex of $V(D) - S$ has at least one in-neighbor in $S$. The domination number of $D$, denoted by $\gamma(D)$, is the cardinality of a smallest dominating set. Denote by $\gamma(D)$-set we mean a dominating set of $D$ with cardinality $\gamma(D)$. In fact, domination in digraphs has many applications such as answering skyline query in database [10] and routing in networks [22]. For more results on domination in digraphs, we refer to [1, 2, 12, 18].

The study of signed domination of undirected graphs was initiated by Dunbar et al. in [5] and continued in [4,6,15,20] and elsewhere. In [23] Zelinka generalizes this concept to digraphs. For a two-valued function $f : V(D) \to \{-1, 1\}$, the *weight* of $f$ is $w(f) = \sum_{v \in V(D)} f(v)$. Formally, a two-valued function $f : V(D) \to \{-1, 1\}$ is said to be a *signed dominating function* if $f(N^-[v]) \ge 1$ for each vertex $v \in V(D)$. The *signed domination number*, denoted by $\gamma_s(D)$, of $D$ is the minimum weight of a signed dominating function on $D$. We call a signed dominating function of weight $\gamma_s(D)$ a $\gamma_s(D)$-function on $D$. Signed domination of digraphs was studied by several authors including [11, 21]. Throughout this paper, if $f$ is a signed dominating function of $D$, then we let $P$ and $M$ denote the sets of those vertices in $D$ which are assigned under $f$ the value 1 and $-1$, respectively. Therefore $|V(D)| = |P| + |M|$ and $\gamma_s(D) = |P| - |M|$.

Product graphs are considered in order to gain global information from the factor graphs [8]. Many interesting wireless networks are based on product graphs with simple factors, such as paths and cycles. In particular, any square grid (resp., torus) is the Cartesian product of two paths (resp., cycles) and any octagonal grid (resp., torus) is the strong product of two paths (resp., cycles) [9]. Recently, the domination numbers of Cartesian product of two directed paths (resp., cycles) have been determined [13, 14, 16, 17, 19, 24].

In this paper, we are going to study domination and signed domination on strong product of two directed paths. We obtain the exact values of $\gamma(\vec{P}_m \boxtimes \vec{P}_n)$ and $\gamma_s(\vec{P}_m \boxtimes \vec{P}_n)$ for any $m$ and $n$, respectively.

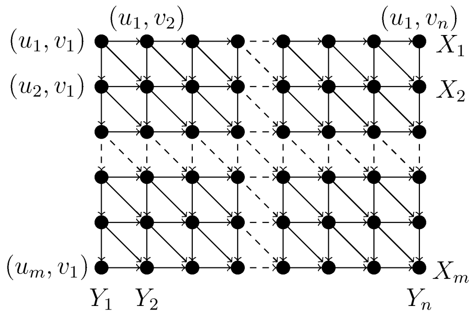


Figure 1: The strong product $\vec{P}_m \boxtimes \vec{P}_n$

## 2 Domination number of $\vec{P}_m \boxtimes \vec{P}_n$

In this section exact values of domination number are determined for strong product $\vec{P}_m \boxtimes \vec{P}_n$ for any $m$ and $n$. Notice that if $m = 1$, then $\vec{P}_m \boxtimes \vec{P}_n$ is isomorphic to $\vec{P}_n$. By the definition of dominating set, the following lemma is easily established.

**Lemma 1** *Let $\vec{P}_n$ be a directed path. Then $\gamma(\vec{P}_n) = \lceil \frac{n}{2} \rceil$.*

**Lemma 2** *Let $S$ be a dominating set of $\vec{P}_m \boxtimes \vec{P}_n$, where $n \geq 2$ . Then $|S \cap (Y_{n-1} \cup Y_n)| \geq \lceil m/2 \rceil$.*

**Proof.** Since each vertex in $Y_{n-1} \cup Y_n$ dominates at most 2 vertices of $Y_n$, it follows that $2|S \cap (Y_{n-1} \cup Y_n)| \geq m$, i.e., $|S \cap (Y_{n-1} \cup Y_n)| \geq \lceil m/2 \rceil$. □

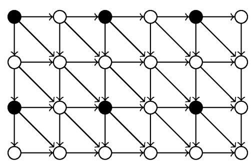

Figure 2: $A$ is the set of bold vertices in $\vec{P}_4 \boxtimes \vec{P}_6$

**Lemma 3** *For any positive integers $k$ and $l$, $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l}) = kl$.*

**Proof.** Let $A_{2i-1} = \cup_{t=1}^{k} (u_{2t-1}, v_{2i-1})$ for $i = 1, 2, \ldots, l$. Then it is easy to verify that $A = \cup_{i=1}^{l} A_{2i-1}$ is a dominating set of $\vec{P}_{2k} \boxtimes \vec{P}_{2l}$ (The dominating set $A$ of $\vec{P}_4 \boxtimes \vec{P}_6$ is shown in Figure 2). So $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l}) \leq |A| = kl$. Let $S$ be a $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l})$-set. Since each vertex in $S$ dominates at most 4 vertices of $\vec{P}_{2k} \boxtimes \vec{P}_{2l}$, we have $4|S| \geq 4kl$, that is, $|S| \geq kl$. Therefore, $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l}) = kl$. □

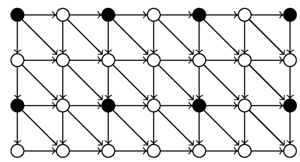

Figure 3: $B$ is the set of bold vertices in $\vec{P}_4 \boxtimes \vec{P}_7$

**Lemma 4** *For any integers $k \geq 1$ and $l \geq 0$, $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) = k(l+1)$.*

**Proof.** We use induction on $l$ to complete the proof of Lemma 4. If $l = 0$, then $\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}$ is isomorphic to $\vec{P}_{2k}$. By Lemma 1, $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) = \gamma(\vec{P}_{2k}) = k$. Assume, then, that $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2r+1}) = k(r+1)$ for all integers $0 \leq r < l$. Now we show that $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) = k(l+1)$. Set $B_{2i-1} = \cup_{t=1}^{k}(u_{2t-1}, v_{2i-1})$ for $i = 1, 2, \ldots, l+1$ and $B = \cup_{i=1}^{l+1} B_{2i-1}$. Then it is not hard to check that $B$ is a dominating set of $\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}$ (The dominating set $B$ of $\vec{P}_4 \boxtimes \vec{P}_7$ is illustrated in Figure 3). Thus $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) \leq |B| = k(l+1)$.

Suppose that $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) = k(l+1)$ is false. Then $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) \leq k(l+1)-1$. Let $S$ be a $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1})$-set. Define $D_1 = D[V(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) \setminus \{Y_{2l}, Y_{2l+1}\}]$. Then $D_1 \cong \vec{P}_{2k} \boxtimes \vec{P}_{2(l-1)+1}$, and so $\gamma(D_1) = \gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2(l-1)+1}) = kl$ by the induction hypothesis. Clearly, $S \cap V(D_1)$ is a dominating set of $D_1$. According to Lemma 2, $\gamma(D_1) \leq |S \cap V(D_1)| = |S| - |S \cap \{Y_{2l}, Y_{2l+1}\}| \leq k(l+1) - 1 - k = kl - 1$, a contradiction. Therefore, $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) = k(l+1)$. $\square$

Notice that $\vec{P}_{2k+1} \boxtimes \vec{P}_{2l} \cong \vec{P}_{2l} \boxtimes \vec{P}_{2k+1}$. We immediately have the next result by Lemma 4.

**Lemma 5** *For any integers $k \geq 0$ and $l \geq 1$, $\gamma(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l}) = l(k+1)$.*

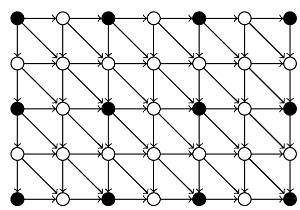

Figure 4: $C$ is the set of bold vertices in $\vec{P}_5 \boxtimes \vec{P}_7$

**Lemma 6** *For any integers $k \geq 0$ and $l \geq 0$, $\gamma(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) = (k+1)(l+1)$.*

**Proof.** We proceed our proof by induction on $l$. Note that $\vec{P}_{2k+1} \boxtimes \vec{P}_1 \cong \vec{P}_{2k+1}$. Thus the assertion is true for $l = 0$ by Lemma 1. Assume that $\gamma(\vec{P}_{2k+1} \boxtimes \vec{P}_{2r+1}) = (k+1)(r+$

1) for all integers $0 \le r < l$. Now we shall prove that $\gamma(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) = (k+1)(l+1)$. Let $C_{2i-1} = \cup_{t=1}^{k+1}(u_{2t-1}, v_{2i-1})$ for $i = 1, 2, \ldots, l+1$. Then one can easily verify that $C = \cup_{i=1}^{l+1} C_{2i-1}$ is a dominating set of $\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}$ (The dominating set $C$ of $\vec{P}_5 \boxtimes \vec{P}_7$ is depicted in Figure 4). So $\gamma(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) \le |C| = (k+1)(l+1)$.

Suppose that $\gamma(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) = (k+1)(l+1)$ is not true. Then $\gamma(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) \le (k+1)(l+1)-1$. Let $S$ be a $\gamma(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1})$-set. Define $D_1 = D[V(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) \setminus \{Y_{2l}, Y_{2l+1}\}]$. Thus $D_1 \cong \vec{P}_{2k+1} \boxtimes \vec{P}_{2(l-1)+1}$, and so $\gamma(D_1) = \gamma(\vec{P}_{2k+1} \boxtimes \vec{P}_{2(l-1)+1}) = (k+1)l$ by the induction hypothesis. Obviously, $S \cap V(D_1)$ is a dominating set of $D_1$. By Lemma 2, $\gamma(D_1) \le |S \cap V(D_1)| = |S| - |S \cap \{Y_{2l}, Y_{2l+1}\}| \le (k+1)(l+1)-1-(k+1) = (k+1)l-1$, which is absurd. Consequently, $\gamma(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) = (k+1)(l+1)$. □

Combining Lemma 1, 3, 4, 5 and 6, we obtain the following theorem.

**Theorem 7** *For any positive integers $m$ and $n$, $\gamma(\vec{P}_m \boxtimes \vec{P}_n) = \lceil m/2 \rceil \lceil n/2 \rceil$.*

## 3 Signed domination number of $\vec{P}_m \boxtimes \vec{P}_n$

Now let us turn our attention to the signed domination number of strong product $\vec{P}_m \boxtimes \vec{P}_n$. We determine the exact values of signed domination number of strong product $\vec{P}_m \boxtimes \vec{P}_n$ for any $m$ and $n$. Recall that $\vec{P}_1 \boxtimes \vec{P}_n \cong \vec{P}_n$. The following three lemmas are trivial from the definition of signed dominating function.

**Lemma 8** *Let $f$ be a signed dominating function of $\vec{P}_m \boxtimes \vec{P}_n$. For every vertex $(u_i, v_j) \in V(\vec{P}_m \boxtimes \vec{P}_n)$, $|N^-[(u_i, v_j)] \cap M| \le 1$, where $2 \le i \le m$ and $2 \le j \le n$ .*

**Lemma 9** *Let $D = (V(D), A(D))$ be a digraph. Then $\gamma_s(D)$ has the same parity as $|V(D)|$.*

**Lemma 10** *Let $\vec{P}_n$ be a directed path. Then $\gamma_s(\vec{P}_n) = n$.*

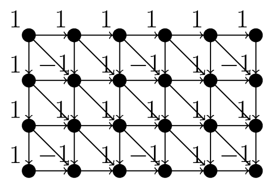

Figure 5: The signed dominating function $f$ of $\vec{P}_4 \boxtimes \vec{P}_6$

**Lemma 11** *For any positive integers $k$ and $l$, $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2l}) = 2kl$.*

**Proof.** Let $F_{2i} = \cup_{t=1}^{k}(u_{2t}, v_{2i})$ for $i = 1, 2, \ldots, l$. Define $f : V(\vec{P}_{2k} \boxtimes \vec{P}_{2l}) \to \{-1, 1\}$ by assigning to each vertex of $\cup_{i=1}^{l} F_{2i}$ the value $-1$ while to each vertex of $V(\vec{P}_{2k} \boxtimes \vec{P}_{2l}) - \cup_{i=1}^{l} F_{2i}$ the value 1. It is not difficult to check that $f$ is a signed dominating function on $\vec{P}_{2k} \boxtimes \vec{P}_{2l}$ with weight $w(f) = 2kl$ (The signed dominating function $f$ on $\vec{P}_4 \boxtimes \vec{P}_6$ is shown in Figure 5). So $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2l}) \le w(f) = 2kl$. By Lemma 8, $|M| = \sum_{j=1}^{l}\sum_{i=1}^{k} |N^-[(u_{2i}, v_{2j})] \cap M| \le kl$. Then $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2l}) = |V(\vec{P}_{2k} \boxtimes \vec{P}_{2l})| - 2|M| \ge 4kl - 2kl = 2kl$. Therefore, $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2l}) = 2kl$. □

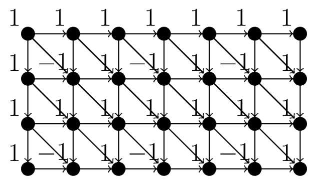


Figure 6: The signed dominating function $f$ of $\vec{P}_4 \boxtimes \vec{P}_7$

**Lemma 12** *For any integers $k \ge 1$ and $l \ge 0$, $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) = 2k(l+1)$.*

**Proof.** The proof is by induction on $l$. If $l = 0$, then $\vec{P}_{2k} \boxtimes \vec{P}_1$ is isomorphic to $\vec{P}_{2k}$. By Lemma 10, $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_1) = \gamma_s(\vec{P}_{2k}) = 2k$. Assume that $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2r+1}) = 2k(r+1)$ for all integers $0 \le r < l$. Now we show that $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) = k(l+1)$. Let $F_{2i}$ be defined as that in the proof of Lemma 11, where $i = 1, 2, \ldots, l$. Define $f : V(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) \to \{-1, 1\}$ as follows: each vertex of $\cup_{i=1}^{l} F_{2i}$ is assigned the value $-1$ while each vertex of $V(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) - \cup_{i=1}^{l} F_{2i}$ is assigned the value 1. It is easily verified that $f$ is a signed dominating function of $\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}$ with weight $w(f) = 2k(l+1)$ (The signed dominating function $f$ of $\vec{P}_4 \boxtimes \vec{P}_7$ is illustrated in Figure 6). Thus $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) \le w(f) = 2k(l+1)$.

Suppose that $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) = 2k(l+1)$ is false. Then $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) \le 2k(l+1) - 2$ by Lemma 9. Let $g$ be a $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1})$-function. Thus $w(g) = |P| - |M| \le 2k(l+1) - 2$. Define $D_1 = D[V(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) \setminus \{Y_{2l}, Y_{2l+1}\}]$. Then $D_1 \cong \vec{P}_{2k} \boxtimes \vec{P}_{2(l-1)+1}$. According to the induction hypothesis, $\gamma_s(D_1) = \gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2(l-1)+1}) = 2kl$. Obviously, $g_1 = g|_{D_1}$ is a signed dominating function of $D_1$. By Lemma 8, we can deduce that $|\{Y_{2l}, Y_{2l+1}\} \cap M| \le k$. So $\gamma_s(D_1) \le w(g_1) = (|P| - 3k) - (|M| - k) \le |P| - |M| - 2k \le 2kl - 2$. This is a contradiction. Therefore, $\gamma_s(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) = 2k(l+1)$. □

Since $\vec{P}_{2k+1} \boxtimes \vec{P}_{2l} \cong \vec{P}_{2l} \boxtimes \vec{P}_{2k+1}$, we immediately obtain the next result by Lemma 12.

**Lemma 13** *For any integers $k \ge 0$ and $l \ge 1$, $\gamma_s(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l}) = 2l(k+1)$.*

**Lemma 14** *For any integers $k \ge 0$ and $l \ge 0$, $\gamma_s(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) = 2(k+1)(l+1) - 1$.*

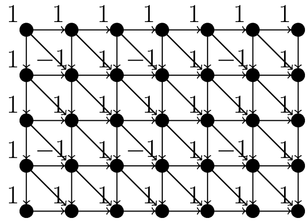


Figure 7: The signed dominating function $f$ of $\vec{P}_5 \boxtimes \vec{P}_7$

**Proof.** If $k = 0$, then $\vec{P}_1 \boxtimes \vec{P}_{2l+1}$ is isomorphic to $\vec{P}_{2l+1}$. The assertion follows directly from Lemma 10. Next we may assume that $k \geq 1$. We proceed our proof by induction on $l$. Recall that $\vec{P}_{2k+1} \boxtimes \vec{P}_1 \cong \vec{P}_{2k+1}$. By Lemma 10, the assertion is trivial for $l = 0$. Now assume that $\gamma_s(\vec{P}_{2k+1} \boxtimes \vec{P}_{2r+1}) = 2(k+1)(r+1)-1$ for all integers $0 \leq r < l$. First we show that $\gamma(\vec{P}_{2k} \boxtimes \vec{P}_{2l+1}) \leq 2(k+1)(r+1)-1$. Let $F_{2i}$ be defined as that in the proof of Lemma 11 for $i = 1, 2, \ldots, l$. Define $f : V(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) \to \{-1, 1\}$ by assigning to each vertex of $\cup_{i=1}^{l} C_{2i}$ the value $-1$ while to each vertex of $V(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) - \cup_{i=1}^{l} C_{2i}$ the value 1. It is not hard to see that $f$ is a signed dominating function of $\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}$ with weight $w(f) = 2(k+1)(l+1) - 1$ (The signed dominating function $f$ of $\vec{P}_5 \boxtimes \vec{P}_7$ is depicted in Figure 7). Hence $\gamma_s(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) \leq w(f) = 2(k+1)(l+1) - 1$.

Suppose that $\gamma_s(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) = 2(k+1)(l+1)-1$ is not true. Then $\gamma_s(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) \leq 2(k+1)(l+1)-3$ by Lemma 9. Let $g$ be a $\gamma_s(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1})$-function. Hence $w(g) = |P| - |M| \leq 2(k+1)(l+1) - 3$. We define $D_1 = D[V(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) \setminus \{Y_{2l}, Y_{2l+1}\}]$. Then $D_1 \cong \vec{P}_{2k+1} \boxtimes \vec{P}_{2(l-1)+1}$, and so $\gamma_s(D_1) = \gamma_s(\vec{P}_{2k+1} \boxtimes \vec{P}_{2(l-1)+1}) = 2l(k+1) - 1$ by the induction hypothesis. Clearly, $g_1 = g|_{D_1}$ is a signed dominating function of $D_1$. Since $X_1 \subseteq P$, we derive that $|\{Y_{2l}, Y_{2l+1}\} \cap M| \leq k$ according to Lemma 8. So $\gamma_s(D_1) \leq w(g_1) = (|P| - 3k - 2) - (|M| - k) \leq |P| - |M| - 2k - 2 \leq 2l(k+1) - 3$, a contradiction. Therefore, $\gamma_s(\vec{P}_{2k+1} \boxtimes \vec{P}_{2l+1}) = 2(k+1)(l+1) - 1$. □

We state the above results as the following theorem.

**Theorem 15** *For any positive integers $m$ and $n$,*

$$\gamma_s(\vec{P}_m \boxtimes \vec{P}_n) = \begin{cases} m\left\lceil \frac{n}{2} \right\rceil & \text{for } m \text{ even,} \\ n\left\lceil \frac{m}{2} \right\rceil & \text{for } m \text{ odd and } n \text{ even,} \\ n\left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor & \text{for } m \text{ odd and } n \text{ odd.} \end{cases}$$

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